Soft Computing (2020) 24:3753–3770
https://doi.org/10.1007/s00500-019-04144-7

METHODOLOGIES AND APPLICATION

Multi-level cognitive concept learning method oriented to data sets with fuzziness: a perspective from features

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Published online: 29 June 2019

Abstract

As a new interdisciplinary field induced by formal concept analysis, rough set, granular computing and cognitive computing, cognitive concept learning has received a great attention in recent years. Cognitive concept learning refers to the acquisition of specific concepts through specific cognitive concept learning approaches. The processes of concept learning mainly focus on simulating human brain recognizing concepts through the modeling of brain intelligence. In this paper, we investigate the mechanism of multi-level cognitive concept learning method oriented to data sets with fuzziness by discussing the process of human cognition. Through a newly defined fuzzy focal feature set, we put forward a corresponding structure of feature-oriented multi-level cognitive concept learning method in data sets with fuzziness from a perspective of philosophical and psychological views of human cognition. To make the presented cognitive concept learning approach much easier to understand and to apply it to practice widely, we establish an algorithm to recognize fuzzy concepts and incomplete fuzzy concepts. In addition, we present a case study about how to recognize and distinguish any two different micro-expressions from an information system with quantitative description to use our proposed method and theory to solve conceptual cognition problems, and also we perform an experimental evaluation on five data sets downloaded from the University of California-Irvine databases. Compared with the existing *granular computing approach to two-way learning*, we obtain more concepts than the two-way learning approach, which shows the feasibility and effectiveness of our feature-oriented multi-level cognitive learning method in data sets with fuzziness.

Keywords Cognitive concept learning · Fuzzy concept · Fuzzy set · Granular computing · Multi-level cognitive

1 Introduction

Derived from the artificial intelligence of a computer system, cognitive computing represents a new computing model that contains a large number of technological innovations on information analysis, natural language processing and machine learning by simulating the human brain (Wang 2009). One goal of cognitive computing is to let the computing system learn, think, and help decision makers to reveal extraordinary insights from large volumes of unstructured data and then make the right decisions like the human brain. Cognitive computing attempts to address inaccurate, uncertain and partially real problems in biological systems to achieve varying degrees of perception, memory, learning,

Communicated by V. Loia.

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language, thinking and problem solving. Based on its own data of the cognitive system, cognitive computing is able to continue self-improving (Modha et al. 2011). At present, with the development of science and technology, and the arrival of large data age, how to know more meaningful knowledge (for example: concept) from vast amounts of information is an urgent need.

Formal concept analysis theory (Ganter and Wille 2012; Wille 1992, 2009) is an area of applied mathematics, and it mainly focuses on cognitive concept based on the mathematization of concept and conceptual hierarchy in a certain formal context. As the central notion in formal concept analysis, concept lattice theory has been researched by most academics, and also formal concept analysis theory and concept lattice theory have been applied to inducing decision trees (Belohlávek et al. 2009), attribute reduction (Konecny 2017; Liu et al. 2007; Pei and Mi 2011; Qi et al. 2015; Wang and Zhang 2008) and the corresponding rule mining (Li et al. 2012, 2016). A formal context is an important part of the theory of formal concept analysis, which is used to express and record the relationship between objects and attributes. Generally speaking, a formal context (Ganter and Wille 2012) is a triple which is composed of an object set, an attribute set and a binary relation between object set and attribute set. In this sense, concept mainly includes two parts: extension (its meaning: an object set) and intension (scope of application: an attribute set), which can be determined from each other (Duntsch and Gediga 2002; Li et al. 2013, 2017; Luksch and Wille 1991; Qi et al. 2014, 2015; Wang 2008; Wille 2009; Yao 2004a, b). Cognitive concept learning refers to acquiring a particular concept by using a specific cognitive learning method. In order to obtain different concepts under different knowledge in practice, the top priority of cognitive concept learning is to establish different cognitive learning methods for reference. So far, different kinds of concepts are put forward to meet different requirements of the problem analysis in practice, which including but not limited to abstract concept (Wang 2008), Wille's concept (Wille 2009), property-oriented concept (Duntsch and Gediga 2002), object-oriented concept (Yao 2004a, b), approximate concept (Li et al. 2013) and three-way concept (Huang et al. 2017; Li et al. 2017; Qi et al. 2014, 2015; Rodríguez-Jiménez et al. 2016; Shivhare and Cherukuri 2017). Nowadays, Li et al. (2015) studied concept learning from the cognitive viewpoint via granular computing. Kumar et al. (2015) made use of formal concept analysis to represent memories and to perform some of the cognitive functions of human brain. In addition, Shivhare et al. (2017) established cognitive relations between the pair of objects and attributes by integrating the idea of cognitive informatics. Zhao et al. (2017) studied the cognitive concept learning from incomplete information. In Moreton et al. (2017), linguistic and non-linguistic pattern learning have been studied separately, and then, the authors provided a comparative approach between them. These concepts can be distinguished from one another based on their intensions and actual demand. In addition, fuzzy concepts have been researched nowadays. Singh and Kumar (2014a, b) proposed a method for generating the bipolar fuzzy formal concepts and a method for decomposition of bipolar fuzzy formal context. Later, Singh (2017) provided an approach to generate the three-way formal fuzzy concepts using the properties of neutrosophic logic and componentwise *Gödel* logic. In our real life, the analysis of n-valued data based on its acceptation, rejection and uncertain part is a major issue. To solve this issue, Singh proposed three methods to discover some useful and interesting information from the three-way n-valued neutrosophic context (Singh 2018e). The first method provided a way to discover n-valued neutrosophic concepts and their concept lattice visualization; the second method provided a way to find some of the closest n-valued concepts at defined threshold. The third method given a multiple ways to zoom in and zoom out the given n-valued neutrosophic contexts at micro- and macrolevel. Furthermore, Singh analyzed the medica data set using the properties of single-valued neutrosophic graph-based concept lattice, and he also proposed a method to select some of the interesting three-way fuzzy concepts at user-defined granulation for their computed Euclidean distance (Singh 2018d). Recently, to process data sets containing vague attributes, Singh proposed a concept learning method by generating vague concept lattice (Singh 2018b). To find some of the hidden or interested pattern from the given m-polar fuzzy context, Singh generalized the mathematical background of concept lattice with m-polar fuzzy sets and its graphical properties (Singh 2018a). To solve the problem of adequate understanding of meaningful pattern existing in bipolar fuzzy concept lattice becomes complex when its size becomes exponential, Singh proposed two methods based on the properties of next neighbors and Euclidean distance (Singh 2019). Moreover, the graphical structure visualization of multi-polar fuzzy contexts was considered as one of the generalization of fuzzy concept lattice beyond the unipolar space (Singh 2018c).

In order to improve the efficiency of cognitive concept learning, the viewpoint of granular computing is particularly worth mentioning. Pedrycz (2008) investigated hybrid methods and models of granular computing based on the foundations of granular computing. Yao (2001, 2012) proposed the formal and mathematical modeling of rule mining based on granular computing. Also, Yao examined a conceptual framework for concept learning from the viewpoints of cognitive informatics and granular computing. Then within this framework, he interpreted concept learning based on a layered model of knowledge discovery (Yao 2009). In fact, information granule is the basic unit of granular computing. Bargiela and Pedrycz (2006) examined the basic motivation for information granulation and cast granular computing as a structured combination of algorithmic and non-algorithmic information processing that mimics human, intelligent synthesis of knowledge from information. Ever since the presentation of information granule, formal concept analysis has grown by leaps and bounds over these years. It has developed tight connection with knowledge discovery, data analysis and visualization as an effective and powerful mathematical tool. Formal concept analysis theory has established its presence in academic and practice. For example, Wu et al. (2009) examined granular structure of concept lattices with application in knowledge reduction in formal concept analysis. Xu et al. (2014) established a novel cognitive system based on formal concept analysis, and then, necessary, sufficient, sufficient and necessary information granules are addressed to exactly describe the human cognitive processes. Furthermore, by using formal concept description of information granules, Xu and Li (2016) proposed a novel granular computing method of machine learning and trained an arbitrary fuzzy information granule to become necessary, sufficient, or necessary and sufficient fuzzy information granule. Li et al. (2015) discussed concept learning via granular computing from the point of view of cognitive computing.

Based on the human recognizing psychology and philosophy, the process of human recognizing a new entity is layer by layer and has a hierarchical structure. However, those well-defined concepts do not reflect the hierarchical structure of human cognition. Furthermore, fuzzy set is more general than classical one. In addition, when we learn something in real life, there are three problems:

(1) Sometimes our aim is to make clear what is the concept (that is to say, what characteristics certain concept has), without understanding the whole concept lattice.
(2) Another important thing in our real life is how to distinguish one object from the other. For example, if we have two animals (a duck and a chicken), then how can we distinguish the duck from the chicken? What characteristics do we need to differentiate the duck from the chicken?
(3) Moreover, in real life, not all of the issues in cognitive concept learning question could have a quantitative description. In other words, there may be a qualitative description of a problem. In dealing with qualitative problems, we first need to quantify qualitative questions (digitalization on non-digitalized problems), and then, we can draw conclusions after reasoning analysis of the qualitative problem. $Example$ 1 : In the investigation of 2017 college graduate employment problems, there may only be two cases: find a job or not find a job. With statistical results, one can quantify "find a job" and "not find a job" to "1" and "0," respectively. We may also encounter the situation of having more than two cases. $Example$ 2 : In the evaluation of teaching hardware facilities of a number of universities, hardware equipment may be artificially evaluated as "poor," "general," "good," "very good" and others. If one sets the first one value, the first two values, the first three values, or the first four values to be "1," and others to be "0," then it will cause information loss problem, which has a significant negative impact on the results of the research. So how to translate qualitative description to quantitative description, and then recognize concepts has certain research significance.

There are two basic ways for human beings to understand the world. One is called "experience + intuition." In ordinary words, after gaining a lot of experiences, they understand a truth and then verify it in new experience. In the past, Confucianism, Buddhism, Mohism and Taoism all used this method. Ancient Greece created another way of knowing the world, a formal logic method. The simplest form of formal logic method is science, which simplifies complex things into several theorems, formulas, and then uses these theorems and formulas to infer to the entire system. However, we usually use the following method: We use the experience and intuition to perceive unknown objects step by step and then construct the process of cognitive concept through logical methods to achieve the purpose of cognition. Based on the analysis above, we need to focus on the process of human cognition and then define a multi-level fuzzy concept which results in a representation of hierarchical structure of human cognition.

So far, none of the existing cognitive concept learning methods could obtain the following results: Obtain all the fuzzy concepts or incomplete fuzzy concepts in data set (to obtain fuzzy concepts); recognize certain objects in each data set (to achieve the purpose of cognition); distinguish between two objects in different object sets of fuzzy concepts or incomplete fuzzy concepts (to achieve the purpose of a deep understanding of the fuzzy concepts). The only related method is found in reference (Xu and Li 2016), and authors could obtain the necessary and sufficient fuzzy information granules (fuzzy concepts in our paper). As for the well-known fuzzy cognitive maps (Kosko 1986), it is proposed by Kosko by fusion Zadeh's fuzzy set theory and Axelrod's cognitive maps theory. Both of the conceptual values and the weights of arc can be fuzzy values in fuzzy cognitive maps. It is a kind of soft computing, which is the product of the combination of fuzzy logic and neural network, and its knowledge representation and reasoning ability are stronger. Both of concepts in fuzzy cognitive maps (Kosko 1986) and fuzzy gray cognitive maps (Salmeron 2010) can be a word, a number and other things. That is to say the concept in fuzzy cognitive maps and fuzzy gray cognitive maps is different with the fuzzy concept (with two parts: objects set and features set) in our paper. Therefore, we cannot make a comparison with fuzzy cognitive maps or fuzzy gray cognitive maps. So we could only make a comparison with the existing two-way learning approach and cannot include the comparison with other methods (including the fuzzy cognitive maps and fuzzy gray cognitive maps). In this paper, we simulate human cognitive process based on the human cognitive psychology and philosophy, and then, we establish a multi-level cognitive concept learning method oriented to data sets with fuzziness from the perspective of features. The biggest contribution of this paper is to deal with the concept learning in data sets with fuzziness, especially in the qualitative description problems. The aims of our feature-oriented multi-level cognitive concept learning method in data sets with fuzziness are:

(1) To obtain all the fuzzy concepts or incomplete fuzzy concepts in data set;
(2) To recognize certain objects in each data set;
(3) To distinguish between two objects (in different object sets of fuzzy concepts or incomplete fuzzy concepts).

It is worth noting that the related (fuzzy) concept in this section is actually a complete (fuzzy) concept in this paper. The remainder is organized as follows. Some basic notions and corresponding properties in fuzzy information system are reviewed in Sect. 2, and complete fuzzy concept and incomplete fuzzy concept are defined. In Sect. 3, we propose a fuzzy focal feature set in a fuzzy information system, and then, some properties of this special fuzzy feature set are discussed. In Sect. 4, we discuss human cognition process and investigate the feature-oriented multi-level cognition mechanism; then, a feature-oriented multi-level cognitive concept learning structure in data sets with fuzziness is constructed based on the cognitive psychology and philosophy. The corresponding program flowchart and pseudocode are proposed to test the effectiveness of our method. In Sect. 5, an illustrative example is presented, and then, our method is used to understand the micro-expressions, and the technology of how to distinguish any two different micro-expressions are also analyzed. In Sect. 6, in order to test our multi-level cognitive method and to make a comparative analysis with the existing granular computing approach to two-way learning, we study the performance of an experiment on five data sets from UCI databases. In the last section, we conclude this paper and explore the future research work.

2 Preliminaries

Throughout this paper, we assume that the universe U is a non-empty finite set, and the class of all subsets of U is denoted by $\mathcal{P}(U)$, the class of all fuzzy subsets of U is denoted by $\mathcal{F}(U)$, and the complementary set of X is denoted by X^c. In this section, we review some basic notions such as fuzzy set, and formal concept in fuzzy formal concept analysis theory from the perspective of the cognitive psychologists and a common sense. Detailed information can be found in the references.

A fuzzy set $\widetilde{A}$ of U is defined as a function assigning to each element x of U, which is introduced by Zadeh (1965). The value $\widetilde{A}(x) \in [0, 1]$ and $\widetilde{A}(x)$ is referred to as the membership degree of x to the fuzzy set $\widetilde{A}$, where $\widetilde{A} : U \to [0, 1]$. Let the class of all fuzzy subsets of U denoted by $\mathcal{F}(U)$. For any fuzzy set $\widetilde{A}, \widetilde{B} \in \mathcal{F}(U)$, we say that $\widetilde{A}$ is contained in $\widetilde{B}$, denoted by $\widetilde{A} \subseteq \widetilde{B}$, if $\widetilde{A}(x) \leq \widetilde{B}(x)$ for all $x \in U$. We say that $\widetilde{A} = \widetilde{B}$ if and only if $\widetilde{A} \subseteq \widetilde{B}$ and $\widetilde{A} \supseteq \widetilde{B}$. The basic computing rules of fuzzy set are described as follows:

$$(\widetilde{A} \cup \widetilde{B})(x) = \max\{\widetilde{A}(x), \widetilde{B}(x)\} = \widetilde{A}(x) \vee \widetilde{B}(x); \tag{1}$$

$$(\widetilde{A} \cap \widetilde{B})(x) = \min\{\widetilde{A}(x), \widetilde{B}(x)\} = \widetilde{A}(x) \wedge \widetilde{B}(x); \tag{2}$$

$$\widetilde{A}^c(x) = 1 - \widetilde{A}(x). \tag{3}$$

Formal context is a triple composed of an object set, an attribute set and a binary relation between object set and attribute set. Attributes express the characteristics of various objects, and the relationship between the attributes expresses the relationship between the concepts in a certain problem. Normally, a formal context is represented by a matrix, and we call this matrix a relation matrix. Each row in the relation matrix represents an object, and each column represents an attribute.

We all know that the relation matrix is a Boolean matrix in the classical formal context. When the Boolean matrix R is degenerated into a fuzzy matrix $\widetilde{R}$, the classical formal context will be degenerated into a fuzzy formal context. Specific definition of fuzzy formal context is presented as follows.

A triple $\widetilde{S} = (U, AT, \widetilde{R})$ is called a fuzzy formal context, where

- $U = \{x_1, x_2, \ldots, x_n\}$ is a non-empty finite set of objects;
- $AT = \{a_1, a_2, \ldots, a_m\}$ is a non-empty finite set of features (also called attributes);
- $\widetilde{R}$ is a fuzzy binary relation of U and AT (i.e., $\widetilde{R} : U \times AT \to [0, 1]$).

It is obvious that the fuzzy formal context $\widetilde{S} = (U, AT, \widetilde{R})$ is actually a fuzzy information system. In the following, we will study the cognitive concept learning method in fuzzy information system.

In a matrix representing a fuzzy information system, we mainly discuss the membership degree of an object with respect to an attribute, and the value of the object under the attribute is a fuzzy number in the interval [0, 1].

Cognitive psychologists generally choose specific research methods according to the actual situation. Among these methods, the most primitive of the research methods is observation. From the point of view of cognitive psychology, observation is a research method which mainly summarizes the laws of cognitive activity by describing and recording the external performance of subjects (such as language, expression and behavior). It is clear that the ability to observe is inborn. In fact, there are examples of this in real life. For example, children form their own way of doing things by observing the behavior of their parents. In other words, parents are the best teachers of children, which is based on the cognitive psychology research method: observation. As a matter of fact, observation is an ability to study and accept new information, which is very important for cognition and perceiving. Then, human beings can observe and remember the characteristics of a kind of material in the process of cognition based on the cognitive psychology. From this perspective, one defines two notions (incomplete fuzzy formal concept and complete fuzzy formal concept) as follows.

Let $\widetilde{S} = (U, AT, \widetilde{R})$ be a fuzzy information system, for arbitrary $X \in \mathcal{P}(U)$, $\widetilde{A}, \widetilde{AT} \in \mathcal{F}(U)$, $(X, X^{\uparrow})$ and $(\widetilde{A}^{\downarrow}, \widetilde{A})$ are called two incomplete fuzzy concepts, where

$$X^{\uparrow} = \widetilde{AT} = \left\{ < a, \bigwedge_{x \in X} \widetilde{R}(x, a) >| \, a \in AT \right\}, \tag{4}$$

$$\widetilde{A}^{\downarrow} = \left\{ x \mid \widetilde{R}(x, a) \geqslant \widetilde{A}(a), \forall a \in A, x \in U \right\}. \tag{5}$$

If $X^{\uparrow} = \widetilde{A}$ and $\widetilde{A}^{\downarrow} = X$, then $(X, X^{\uparrow}) = (\widetilde{A}^{\downarrow}, \widetilde{A}) = (X, \widetilde{A})$ is called a complete fuzzy concept, or simply called a fuzzy concept.

In a formal context, each concept consists of two parts: extension and intension. Where extension is a set of objects which belong to the concept, intension is a set of attributes shared by the objects in the concept. The semantic interpretation of the definitions of extension and intension is that they are all the entities of a certain property and all the feature descriptions of certain objects, respectively. In a fuzzy information system, for a fuzzy concept $(X, \widetilde{A})$, X is the extension and $\widetilde{A}$ is the intension of fuzzy concept $(X, \widetilde{A})$. $X^{\uparrow}$ is a fuzzy set with respect to feature set AT, and the corresponding membership degree of each feature is the smallest value in each column of X. $\widetilde{A}^{\downarrow}$ is a set of objects, in which the membership degree is not less than the value $\widetilde{A}(a)$.

Let $\widetilde{S} = (U, AT, \widetilde{R})$ be a fuzzy information system, $X_1, X_2, X \in \mathcal{P}(U)$, $A_1, A_2, A \subseteq AT$, and $\widetilde{A_1}, \widetilde{A_2}, \widetilde{A} \in \mathcal{F}(U)$, then the above two operators $\uparrow$ and $\downarrow$ have the following properties.

(1) $X \subseteq X^{\uparrow\downarrow}$, $\widetilde{A} \subseteq \widetilde{A}^{\downarrow\uparrow}$;
(2) $X^{\uparrow} = X^{\uparrow\downarrow\uparrow}$, $\widetilde{A}^{\downarrow} = \widetilde{A}^{\downarrow\uparrow\downarrow}$;
(3) $X \subseteq \widetilde{A}^{\downarrow} \Leftrightarrow \widetilde{A} \subseteq X^{\uparrow}$;
(4) $X_1 \subseteq X_2 \Rightarrow X_2^{\uparrow} \subseteq X_1^{\uparrow}$, $\widetilde{A_1} \subseteq \widetilde{A_2} \Rightarrow \widetilde{A_2}^{\downarrow} \subseteq \widetilde{A_1}^{\downarrow}$;
(5) $(X_1 \cup X_2)^{\uparrow} = X_1^{\uparrow} \cap X_2^{\uparrow}$, $(\widetilde{A_1} \cup \widetilde{A_2})^{\downarrow} = \widetilde{A_1}^{\downarrow} \cap \widetilde{A_2}^{\downarrow}$;
(6) $(X_1 \cap X_2)^{\uparrow} \supseteq X_1^{\uparrow} \cup X_2^{\uparrow}$, $(\widetilde{A_1} \cap \widetilde{A_2})^{\downarrow} \supseteq \widetilde{A_1}^{\downarrow} \cup \widetilde{A_2}^{\downarrow}$;
(7) $(X^{\uparrow\downarrow}, X^{\uparrow})$ and $(\widetilde{A}^{\downarrow}, \widetilde{A}^{\downarrow\uparrow})$ are two fuzzy concepts.

3 A special fuzzy set: fuzzy focal feature set

In order to learn concepts from a fuzzy information system, we need to go into details the research of knowledge in the fuzzy information system. Especially, the relationship between different features expresses the relationship between the concepts of the research question. So finding the relationship between features is an important problem to be solved. Next we will put forward a definition on fuzzy focal feature set in fuzzy formal context, and the corresponding properties are investigated.

Definition 3.1 Let $\widetilde{S} = (U, AT, \widetilde{R})$ be a fuzzy information system, a fuzzy feature set $\widetilde{AT_1}$ is called a fuzzy focal feature set, if for an arbitrary $a \in AT$, $\widetilde{AT_1}(a) = \bigwedge_{x \in U} \widetilde{R}(x, a)$. In other words, the fuzzy focal feature set is

$$\widetilde{AT_1} = \left\{ < a, \bigwedge_{x \in U} \widetilde{R}(x, a) > |a \in AT \right\}. \tag{6}$$

Remark Definition 3.1 can be redefined as follows based on the definitions of operators $\uparrow$ and $\downarrow$ in the last section.

Let $\widetilde{S} = (U, AT, \widetilde{R})$ be a fuzzy information system, then $U^{\uparrow}$ is called the fuzzy focal feature set, since

$$U^{\uparrow} = \left\{ < a, \bigwedge_{x \in U} \widetilde{R}(x, a) > |a \in AT \right\} = \widetilde{AT_1}. \tag{7}$$

In fact, fuzzy focal feature set is a meaningful definition in the cognitive process. Its semantic interpretation is the set of all the minimum feature values in each column with respect to all the objects in universe.

Since the fuzzy focal feature set is a special fuzzy feature set in a fuzzy information system, it satisfies all the properties mentioned in the last section. Next we will discuss some special properties of the fuzzy focal feature set.

Theorem 3.1 *Let $\widetilde{S} = (U, AT, \widetilde{R})$ be a fuzzy information system, then the following two properties are equivalent.*

(1) *$\widetilde{AT_1}$ is the fuzzy focal feature set of $\widetilde{S}$;*
(2) *$(U, \widetilde{AT_1})$ is a fuzzy concept.*

Proof Sufficiency and necessity can be proved straightforwardly based on the definition of fuzzy focal feature set in Definition 3.1 and Remark. □

Table 1 A fuzzy information system

	a_1	a_2	a_3	a_4	a_5
x_1	0.3	0.8	0.7	0.9	0.4
x_2	0.6	0.4	0.3	0.5	0.3
x_3	0.7	0.2	0.8	0.8	0.8
x_4	0.9	0.5	0.6	0.5	0.7

In order to understand the definition of fuzzy focal feature set and its corresponding properties well, we will use an example to illustrate the meaning of the fuzzy focal feature set and its properties.

Example 3.1 Table 1 is a fuzzy information system $\widetilde{S} = (U, AT, \widetilde{R})$, where $U = \{x_1, x_2, x_3, x_4\}$ and $AT = \{a_1, a_2, a_3, a_4, a_5\}$.

From this fuzzy information system, we can calculate

$$
\begin{aligned}
U^{\uparrow} &= \left\{ < a, \bigwedge_{x \in U} \widetilde{R}(x, a) > | a \in AT \right\} \\
&= \{<a_1, 0.3>, <a_2, 0.2>, <a_3, 0.3>, <a_4, 0.5>, <a_5, 0.3>\} \\
&= \widetilde{AT_1}.
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{AT_1}^{\downarrow} &= \{x | \widetilde{R}(x, a) \geqslant \widetilde{AT_1}(a), \forall a \in AT\} \\
&= \left\{ x | \widetilde{R}(x, a) \geqslant \bigwedge_{x \in U} \widetilde{R}(x, a), \forall \in AT \right\} \\
&= \{x_1, x_2, x_3, x_4\} = U.
\end{aligned}
$$

Then from the above results, one obtains the following outcomes directly.
$\widetilde{AT_1}$ is the fuzzy focal feature set of $\widetilde{S}$.
$(U, \widetilde{AT_1}) = (U, \{< a_1, 0.3 >, < a_2, 0.2 >, < a_3, 0.3 >, < a_4, 0.5 >, < a_5, 0.3 >\})$ is an obvious fuzzy concept.

4 Feature-oriented multi-level cognitive concept learning in fuzzy data

Based on the defined special fuzzy set in the last section, we will construct an feature-oriented multi-level cognitive concept learning method by using fuzzy focal feature set under the guidance of the philosophical principle of human cognition in this section. The basic principle of human cognition is to find different things from the same, so as to accomplish the task of cognition. One of the fundamental human cognitive processes is problem solving (Wang and Chiew 2010). The process of cognition or solving problem is actually converting the knowledge of unknown to the knowledge of known. For the knowledge completely unknown, one can learn the knowledge through study. In fact, one could achieve the goal of converting the knowledge of unknown into the knowledge of known step by step through some transfer functions. That is to say, when the knowledge of unknown is transformed into the knowledge of known, it can be realized by some transfer functions. The choice of transfer function should be decided according to the concrete problem. In other words, the process of human cognition is hierarchical. The human cognitive process can be described in Fig. 1.

In Fig. 1, the blank block is the knowledge of unknown. The block with blue horizontal line is the local knowledge of known by using a transfer function f_1. The block with green left slash, the block with purple right slash and the block with red vertical line are the knowledge of known by using transfer functions f_2, f_3 and f_n, respectively. After nth cognition (from Level 1 to Level n), the knowledge of unknown (in Level 0) can be converted to the knowledge of known (in Level n) by taking advantage of transfer functions $f_1, f_2, f_3, \ldots, f_n$. The purpose of this figure is to simulate human cognitive process, and this is also the overall main idea of our multi-level cognitive concept learning method.

For a certain fuzzy information system and to understand some fuzzy concepts by using a multi-level method, one can consider three trains of thoughts. The first idea is to learn the fuzzy concept gradually from the point of view of features, which can be called a feature-oriented multi-level cognitive concept learning method in fuzzy data. The second approach is to study the fuzzy information system from the perspective of objects, which is called an object-oriented multi-level cognitive concept learning method in fuzzy data. The third technique is mainly to think about the problem from both features and objects point of view (i.e., the whole fuzzy information system), and this pattern can be called a feature-object-oriented multi-level cognitive concept learning method with data sets with fuzziness. In this paper, we

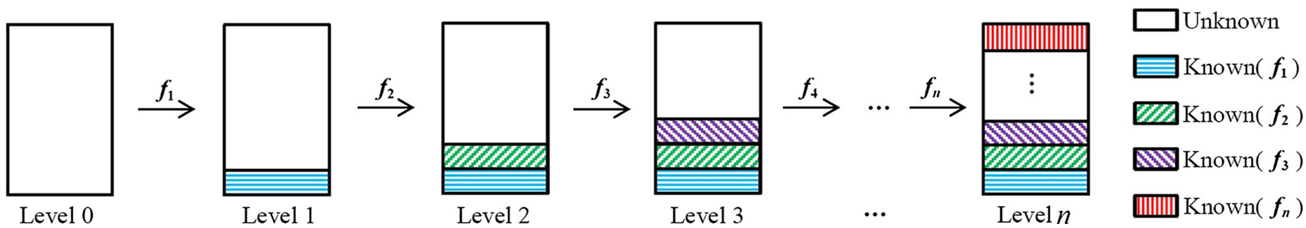

Fig. 1 Simulating human cognitive process

only take the feature-oriented multi-level cognitive concept learning method into account to solve the problem of cognitive concept learning in fuzzy data.

The basic principle of human cognitive is that one can distinguish and recognize objects by using the universality and particularity. If someone wants to distinguish two objects, he/she needs to find out the differences between these two objects. In reality, we may encounter the following situations. If there exist some certain features that can be owned by one of these two objects, one can easily distinguish them. While if we only know the membership degrees (numerical in interval [0,1]) with respect to some certain features of these two objects, how can we distinguish them? Even worse, if the feature values of these two objects are qualitative description, then how to distinguish one object from the other? One of our purposes of writing this paper is to solve these two problems.

In order to give an exact answer to a fuzzy question, it is necessary to introduce the concept of cut-set to non-fuzzify the fuzzy set. For a fuzzy set $\widetilde{A} = \{(u, A(u))|u \in U, A(u) \in [0, 1]\}$, if one knows the threshold λ $(0 \leqslant \lambda \leqslant 1)$, then the λ cut-set of fuzzy set $\widetilde{A}$ is represented as $\widetilde{A}_\lambda = \{u|A(u) \geqslant \lambda, u \in U\}$. With the increase in the threshold λ, the cut-set $\widetilde{A}_\lambda$ becomes smaller, so as to achieve the purpose of distinguishing the objects. For a certain path, with the increasing level in multi-level cognitive concept learning, there is at least an increase in the threshold of one feature. That is to say, the bigger the number of level, the greater the increase in the threshold λ of at least one feature. In fact, this is the main idea of our feature-oriented multi-level cognitive concept learning method in fuzzy data. The detailed method will be discussed in the next.

For a fuzzy information system $\widetilde{S} = (U, AT, \widetilde{R})$, the detailed and completed feature-oriented multi-level cognitive concept learning procedure and method in fuzzy data are presented as follows.

Step 1 Calculate fuzzy focal feature set $\widetilde{AT_1}$, generate a fuzzy concept $(U, \widetilde{AT_1})$ in Level 1.

In this fuzzy information system $\widetilde{S} = (U, AT, \widetilde{R})$, for an arbitrary feature $a \in AT$, compute $\bigwedge\limits_{x\in U} \widetilde{R}(x, a)$, then $\widetilde{AT_1} = \{< a, \bigwedge\limits_{x\in U} \widetilde{R}(x, a) > |a \in AT\}$ is the fuzzy focal feature set of $\widetilde{S}$. Accordingly, generate an incomplete fuzzy concept $(U, \widetilde{AT_1})$. It is clear that $U^\uparrow = \widetilde{AT_1}$ and $\widetilde{AT_1}^\downarrow = U$, so $(U, \widetilde{AT_1})$ is a fuzzy concept in Level 1.

In fact, this step can be described simply as follows:

Compute $U^\uparrow = \widetilde{AT_1}$; then, one gets the fuzzy concept $(U, \widetilde{AT_1}) = (U, U^\uparrow)$ in Level 1, where $\widetilde{AT_1}$ is the fuzzy focal feature set of $\widetilde{S}$.

Step 2 Sort the feature values from small to large for each feature. Calculate and generate fuzzy concepts or incomplete fuzzy concepts in middle levels and the highest level.

For an arbitrary feature $a \in AT$, the feature values can be sorted from small to large: $\widetilde{A}(a)^1 \leqslant \widetilde{A}(a)^2 \leqslant \cdots \leqslant \widetilde{A}(a)^n$, where $\widetilde{A}(a)^1$ and $\widetilde{A}(a)^n$ represent the minimum value and the maximum value of all the feature values with respect to the feature a in the universe U, respectively.

From Fig. 1, one can find that the local knowledge of known in Level i is derived by a transfer function f_i and the corresponding fuzzy concept in Level $i-1$, and the local knowledge of known in Level $i+1$ is also derived by another transfer function f_{i+1} and the corresponding fuzzy concept in Level i, where $i \leqslant m$. Hence, the local knowledge of known in higher level can be derived by a transfer function and the corresponding fuzzy concept in the above level. So, without loss of generality, we can suppose that all the fuzzy concepts and incomplete fuzzy concepts in the above level (i.e., Level i) are $(X_{i1}, \widetilde{AT_{i1}})$, $(X_{i2}, \widetilde{AT_{i2}})$, …, $(X_{iq}, \widetilde{AT_{iq}})$, …, $(X_{ip}, \widetilde{AT_{ip}})$, where $q \leqslant p \leqslant n$, and $\widetilde{AT_{iq}} = \{< a, \bigwedge\limits_{x\in X_{iq}} \widetilde{R}(x, a) > |a \in AT\}$. In order to compute all the fuzzy concepts and incomplete fuzzy concepts in the next level (i.e., Level $i+1$), for definiteness and without loss of generality, we take the fuzzy concept or incomplete fuzzy concept $(X_{iq}, \widetilde{AT_{iq}}) = (X_{iq}, \{< a, \bigwedge\limits_{x\in X_{iq}} \widetilde{R}(x, a) > |a \in AT\})$ in Level i as an example, and we will encounter the five situations list as **Case 1** to **Case 5** in the following.

Case 1 If $|X_{iq}| = 1$, and $(X_{iq}, \widetilde{AT_{iq}})$ satisfies $X_{iq}^\uparrow \neq \widetilde{AT_{iq}}$, then $(X_{iq}, \widetilde{AT_{iq}})$ is an incomplete fuzzy concept in Level i. So the fuzzy concept in Level $i+1$ is $(X_{iq}, X_{iq}^\uparrow)$, and cognitive learning on this path ended.

Case 2 If $|X_{iq}| = 1$, and $(X_{iq}, \widetilde{AT_{iq}})$ satisfies $X_{iq}^\uparrow = \widetilde{AT_{iq}}$, then $(X_{iq}, \widetilde{AT_{iq}})$ is a fuzzy concept in Level i, and cognitive learning on this path ended.

Case 3 If $|X_{iq}| \geqslant 2$, and $(X_{iq}, \widetilde{AT_{iq}})$ satisfies $X_{iq}^\uparrow \neq \widetilde{AT_{iq}}$, then $(X_{iq}, \widetilde{AT_{iq}})$ is an incomplete fuzzy concept in Level i, and the fuzzy concept in Level $i+1$ is $(X_{iq}, X_{iq}^\uparrow)$.

Case 4 If $|X_{iq}| \geqslant 2$, and $(X_{iq}, \widetilde{AT_{iq}})$ satisfies $X_{iq}^\uparrow = \widetilde{AT_{iq}}$, then $(X_{iq}, \widetilde{AT_{iq}})$ is a fuzzy concept in Level i. Furthermore, if there exist two features $a_s, a_t \in AT$ such that

$$\{< a \neq a_s, \widetilde{AT_{iq}}(a) > \vee < a_s, \widetilde{A}(a_s)^f >\}^\downarrow \subsetneqq X_{iq}, \tag{8}$$

$$\{< a \neq a_t, \widetilde{AT_{iq}}(a) > \vee < a_t, \widetilde{A}(a_t)^g >\}^\downarrow \subsetneqq X_{iq}, \tag{9}$$

$$\begin{aligned}&\{< a \neq a_s, \widetilde{AT_{iq}}(a) > \vee < a_s, \widetilde{A}(a_s)^f >\}^\downarrow \\ &\cup\{< a \neq a_t, \widetilde{AT_{iq}}(a) > \vee < a_t, \widetilde{A}(a_t)^g >\}^\downarrow = X_{iq}.\end{aligned} \tag{10}$$

where $\widetilde{A}(a_s)^{f-1} \leqslant \widetilde{AT_{iq}}(a_s) < \widetilde{A}(a_s)^f$ and $\widetilde{A}(a_t)^{g-1} \leqslant \widetilde{AT_{iq}}(a_t) < \widetilde{A}(a_t)^g$, $2 \leqslant f, g \leqslant n$.

If $\{< a \neq a_s, \widetilde{AT_{iq}}(a) > \vee < a_s, \widetilde{A}(a_s)^f >\}^\downarrow$ and $\{< a \neq a_t, \widetilde{AT_{iq}}(a) > \vee < a_t, \widetilde{A}(a_t)^g >\}^\downarrow$ are different with all the object sets in above levels, then we get the fuzzy concepts or incomplete fuzzy concepts in Level $i+1$:

$$(\{< a \neq a_s, \widetilde{AT_{iq}}(a) > \vee < a_s, \widetilde{A}(a_s)^f >\}^{\downarrow}, \{< a \neq a_s, \widetilde{AT_{iq}}(a) > \vee < a_s, \widetilde{A}(a_s)^f >\}), \tag{11}$$

$$(\{< a \neq a_t, \widetilde{AT_{iq}}(a) > \vee < a_t, \widetilde{A}(a_t)^g >\}^{\downarrow}, \{< a \neq a_t, \widetilde{AT_{iq}}(a) > \vee < a_t, \widetilde{A}(a_t)^g >\}). \tag{12}$$

If one of these two object sets (i.e., $\{< a \neq a_s, \widetilde{AT_{iq}}(a) > \vee < a_s, \widetilde{A}(a_s)^f >\}^{\downarrow}$ or $\{< a \neq a_t, \widetilde{AT_{iq}}(a) > \vee < a_t, \widetilde{A}(a_t)^g >\}^{\downarrow}$) is equal to the object set in above levels, then we delete the corresponding fuzzy concept or incomplete fuzzy concept in Level $i+1$.

Case 5 If $|X_{iq}| \geqslant 2$, and $(X_{iq}, \widetilde{AT_{iq}})$ satisfies $X_{iq}^{\uparrow} = \widetilde{AT_{iq}}$, then $(X_{iq}, \widetilde{AT_{iq}})$ is a fuzzy concept in Level i. Furthermore, if there only exists one feature $a_h \in AT$ such that

$$\emptyset \subsetneqq \{< a \neq a_h, \widetilde{AT_{iq}}(a) > \vee < a_h, \widetilde{A}(a_h)^k >\}^{\downarrow} \subsetneqq X_{iq}, \tag{13}$$

$$(X_{iq} - \{< a \neq a_h, \widetilde{AT_{iq}}(a) > \vee < a_h, \widetilde{A}(a_h)^k >\}^{\downarrow})^{\uparrow} = \widetilde{AT_{iq}}. \tag{14}$$

where $\widetilde{A}(a_h)^{k-1} \leqslant \widetilde{AT_{iq}}(a_h) < \widetilde{A}(a_h)^k, 2 \leqslant k \leqslant n$.

If $\{< a \neq a_h, \widetilde{AT_{iq}}(a) > \vee < a_h, \widetilde{A}(a_h)^k >\}^{\downarrow}$ and $X_{iq} - \{< a \neq a_h, \widetilde{AT_{iq}}(a) > \vee < a_h, \widetilde{A}(a_h)^k >\}^{\downarrow}$ are different with all the object sets in above levels, then we get the fuzzy concepts or incomplete fuzzy concepts in Level $i+1$:

$$(\{< a \neq a_h, \widetilde{AT_{iq}}(a) > \vee < a_h, \widetilde{A}(a_h)^k >\}^{\downarrow}, \{< a \neq a_h, \widetilde{AT_{iq}}(a) > \vee < a_h, \widetilde{A}(a_h)^k >\}), \tag{15}$$

$$(X_{iq} - \{< a \neq a_h, \widetilde{AT_{iq}}(a) > \vee < a_h, \widetilde{A}(a_h)^k >\}^{\downarrow}, \widetilde{AT_{iq}}). \tag{16}$$

If one of these two object sets (i.e., $\{< a \neq a_s, \widetilde{AT_{iq}}(a) > \vee < a_s, \widetilde{A}(a_s)^f >\}^{\downarrow}$ or $\{< a \neq a_t, \widetilde{AT_{iq}}(a) > \vee < a_t, \widetilde{A}(a_t)^g >\}^{\downarrow}$) is equal to the object set in above levels, then we delete the corresponding fuzzy concept or incomplete fuzzy concept in Level $i+1$.

One must note that the object set of the fuzzy concept $(X_{iq}, \widetilde{AT_{iq}})$ satisfies $|X_{iq}| = 1$ in **Case 1** and **Case 2**, and it means that the object in set X_{iq} can be studied by us.

Started from calculating the fuzzy focal feature set, one can get all the fuzzy concepts and incomplete fuzzy concepts in Level 2. Then, one can recognize all the fuzzy concepts and incomplete fuzzy concepts in each level by using the five cases mentioned above.

Step 3 Draw the map of the cognitive process of our feature-oriented multi-level cognitive concept learning method in fuzzy data. Start from Level 1 and stop until one cannot distinguish the object set in each fuzzy concept and incomplete fuzzy concept.

After drawn the map of feature-oriented multi-level cognitive concept learning process in fuzzy data, we can distinguish any two objects (in different object sets of fuzzy concepts or incomplete fuzzy concepts) on the basis of the cognitive process. In order to distinguish any two different objects, first we can find a fuzzy concept or an incomplete fuzzy concept in lower level in the cognitive map, in which the (incomplete) fuzzy concept contains both of these two objects. Then, the classification standard of this fuzzy concept or this incomplete fuzzy concept in the next level is the characteristics which can distinguish these two objects. We will introduce the detailed method of how to distinguish between two objects in the case study section.

To understand our method, in the following we provide the corresponding program flowchart and pseudocode of feature-oriented multi-level cognitive concept learning procedure and method in fuzzy data. Figure 2 is the program flowchart of our feature-oriented multi-level cognitive procedure in fuzzy data, and Algorithm 1 is the pseudocode of feature-oriented multi-level cognitive concept learning method in a fuzzy information system, where the representing of A, B, C and D are listed as follows:

- **A** *represents*

$$a_s, a_t \in AT, \text{ and } \{< a \neq a_s, \widetilde{AT}(a) > \vee < a_s, \widetilde{A}(a_s)^f >\}^{\downarrow} \subsetneqq X, \text{ and } \{< a \neq a_t, \widetilde{AT}(a) > \vee < a_t, \widetilde{A}(a_t)^g >\}^{\downarrow} \subsetneqq X, \text{ and } \{< a \neq a_s, \widetilde{AT}(a) > \vee < a_s, \widetilde{A}(a_s)^f >\}^{\downarrow} \cup \{< a \neq a_t, \widetilde{AT}(a) > \vee < a_t, \widetilde{A}(a_t)^g >\}^{\downarrow} = X. \tag{17}$$

- **B** *represents*

$$C\{l\}\{index++\} = (\{< a \neq a_s, \widetilde{AT}(a) > \vee < a_s, \widetilde{A}(a_s)^f >\}^{\downarrow}, \{< a \neq a_s, \widetilde{AT}(a) > \vee < a_s, \widetilde{A}(a_s)^f >\}),$$
$$C\{l\}\{index++\} = (\{< a \neq a_t, \widetilde{AT}(a) > \vee < a_t, \widetilde{A}(a_t)^g >\}^{\downarrow}, \{< a \neq a_t, \widetilde{AT}(a) > \vee < a_t, \widetilde{A}(a_t)^g >\}). \tag{18}$$

- **C** *represents*

$$a_h \in AT, \text{ and } \emptyset \subsetneqq \{< a \neq a_h, \widetilde{AT}(a) > \vee < a_h, \widetilde{A}(a_h)^k >\}^{\downarrow} \subsetneqq X, \tag{19}$$
$$\text{and } (X - \{< a \neq a_h, \widetilde{AT}(a) > \vee < a_h, \widetilde{A}(a_h)^k >\}^{\downarrow})^{\uparrow} = \widetilde{AT}.$$

- **D** *represents*

$$C\{l\}\{index++\} = (\{< a \neq a_h, \widetilde{AT}(a) > \vee < a_h, \widetilde{A}(a_h)^k >\}^{\downarrow}, \{< a \neq a_h, \widetilde{AT}(a) > \vee < a_h, \widetilde{A}(a_h)^k >\}),$$
$$C\{l\}\{index++\} = (X - \{< a \neq a_h, \widetilde{AT}(a) > \vee < a_h, \widetilde{A}(a_h)^k >\}^{\downarrow}, \widetilde{AT}). \tag{20}$$

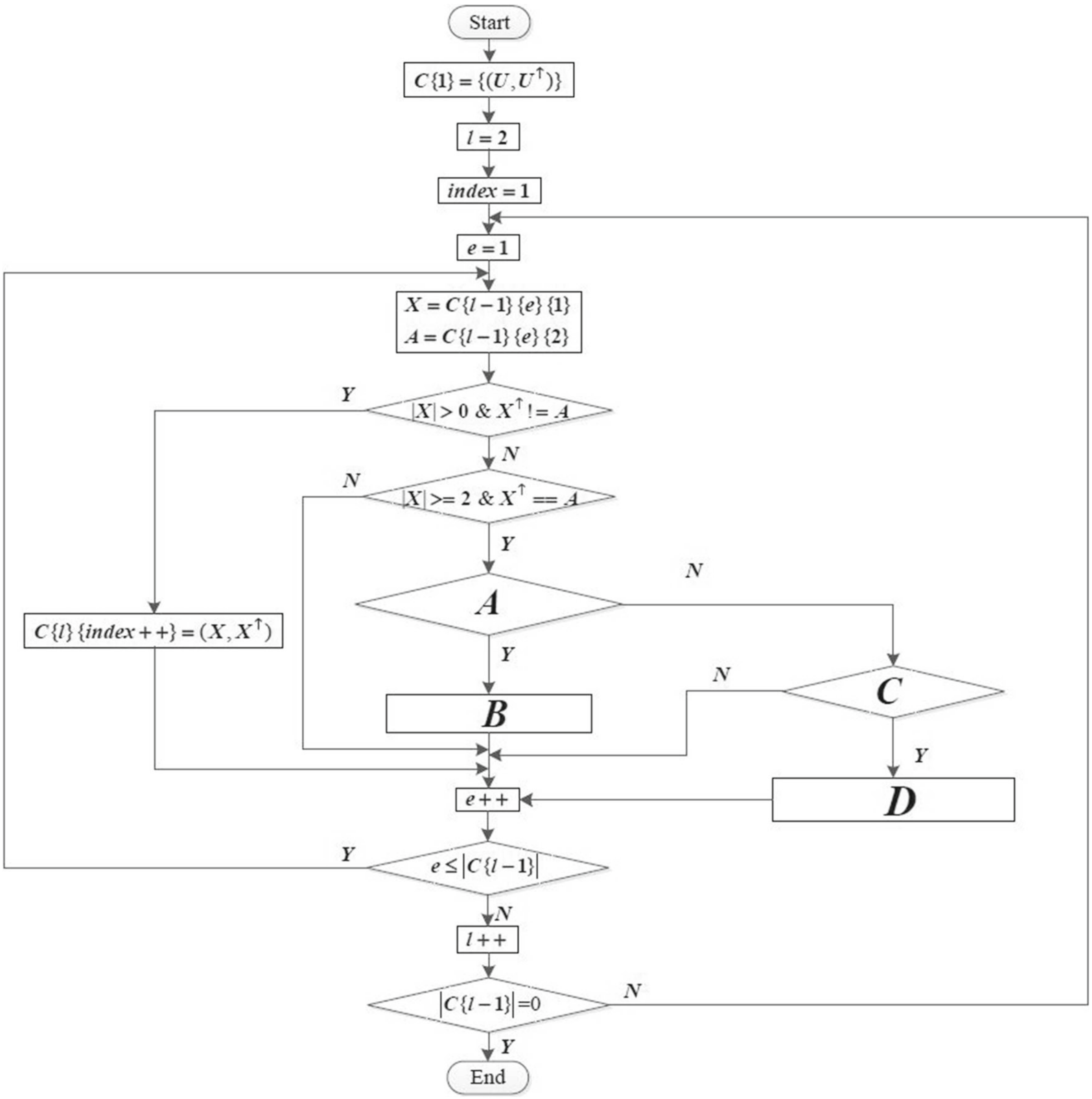

Fig. 2 Program flowchart of feature-oriented multi-level recognize in fuzzy data

Description and explanation of Fig. 2 and Algorithm 1 Input a fuzzy information system $\widetilde{S} = (U, AT, \widetilde{R})$, what we need to do first is to compute the fuzzy focal feature set $\widetilde{AT_1}$, then we can get a fuzzy concept $(U, U^{\uparrow}) = (U, \widetilde{AT_1})$ in Level 1. In order to calculate all the fuzzy concepts and incomplete fuzzy concepts in middle levels and the highest level, we need to find out some features which can distinguish some objects from the others in a certain object set. The most important thing is to judge which kinds of situations they are experiencing. There are five cases for us to judge and choose; then, we can get the fuzzy concepts or incomplete fuzzy concepts in the next level. Through the repeat until loop in this program, one can compute all the fuzzy concepts and incomplete fuzzy concepts in each level. The variable e represents the eth (incomplete) fuzzy concept in the previous level, and $index$ represents the subscript of fuzzy concepts or incomplete fuzzy concepts in this level. $C\{l-1\}$ represents the set of fuzzy concepts and incomplete fuzzy concepts in Level $l-1$. $C\{l-1\}\{e\}\{1\}$ and $C\{l-1\}\{e\}\{2\}$ represent the object set and feature set of the eth fuzzy concept in Level $l-1$, respectively. $\widetilde{A}(a_s)^f$, $\widetilde{A}(a_t)^g$ and $\widetilde{A}(a_h)^k$ are the smallest feature values greater than $\widetilde{AT}(a_s)$, $\widetilde{AT}(a_t)$ and $\widetilde{AT}(a_h)$ with respect to features a_s, a_t and a_h, respectively.

So far, the cognitive process of our multi-level method oriented to features in fuzzy data is completed, and it is obvious that all the fuzzy concepts and incomplete fuzzy concepts can be found after multi-level cognitive, then how to distinguish any two different objects come into question. We will introduce an example to illustrate the utilization of our

Algorithm 1: Feature-oriented multi-level cognitive concept learning method in fuzzy information system

Input : A fuzzy information system $\widetilde{S} = (U, AT, \widetilde{R})$, where $U = \{x_1, x_2, \ldots, x_n\}$, $AT = \{a_1, a_2, \ldots, a_m\}$
Output : All the fuzzy concepts and incomplete fuzzy concepts in each level

begin
  //*Find concept in Level 1*//
  $C\{1\} \leftarrow \{(U, U^{\uparrow})\}$
  $l \leftarrow 2$
  //*Find concepts from Level 2 to Level l, $l \geqslant 2$*//
  $index \leftarrow 1$
  repeat
    for $e = 1$ *to* $|C\{l-1\}|$ **do**
      $X \leftarrow C\{l-1\}\{e\}\{1\}$
      $\widetilde{AT} \leftarrow C\{l-1\}\{e\}\{2\}$
      if $|X| > 0$ *and* $X^{\uparrow} != \widetilde{AT}$ **then**
        $C\{l\}\{index++\} \leftarrow (X, X^{\uparrow})$
      else if $|X| \geqslant 2$ *and* $X^{\uparrow} == \widetilde{AT}$ **then**
        //*Find $a_s, a_t, a_h \in AT$, such that $\widetilde{A}(a_s)^f > \widetilde{AT}(a_s)$, $\widetilde{A}(a_t)^g > \widetilde{AT}(a_t)$, $\widetilde{A}(a_h)^k > \widetilde{AT}(a_h)$*//
        if $\{< a \neq a_s, \widetilde{AT}(a) > \vee < a_s, \widetilde{A}(a_s)^f >\}^{\downarrow} \subsetneqq X$, $\{< a \neq a_t, \widetilde{AT}(a) > \vee < a_t, \widetilde{A}(a_t)^g >\}^{\downarrow} \subsetneqq X$, *and* $\{< a \neq a_s, \widetilde{AT}(a) > \vee < a_s, \widetilde{A}(a_s)^f >\}^{\downarrow} \cup \{< a \neq a_t, \widetilde{AT}(a) > \vee < a_t, \widetilde{A}(a_t)^g >\}^{\downarrow} = X$ **then**
          $C\{l\}\{index++\} \leftarrow (\{< a \neq a_s, \widetilde{AT}(a) > \vee < a_s, \widetilde{A}(a_s)^f >\}^{\downarrow}, \{< a \neq a_s, \widetilde{AT}(a) > \vee < a_s, \widetilde{A}(a_s)^f >\})$
          $C\{l\}\{index++\} \leftarrow (\{< a \neq a_t, \widetilde{AT}(a) > \vee < a_t, \widetilde{A}(a_t)^g >\}^{\downarrow}, \{< a \neq a_t, \widetilde{AT}(a) > \vee < a_t, \widetilde{A}(a_t)^g >\})$
          else if $\emptyset \subsetneqq \{< a \neq a_h, \widetilde{AT}(a) > \vee < a_h, \widetilde{A}(a_h)^k >\}^{\downarrow} \subsetneqq X$ *and* $(X - \{< a \neq a_h, \widetilde{AT}(a) > \vee < a_h, \widetilde{A}(a_h)^k >\}^{\downarrow})^{\uparrow} = \widetilde{AT}$ **then**
            $C\{l\}\{index++\} \leftarrow (\{< a \neq a_h, \widetilde{AT}(a) > \vee < a_h, \widetilde{A}(a_h)^k >\}^{\downarrow}, \{< a \neq a_h, \widetilde{AT}(a) > \vee < a_h, \widetilde{A}(a_h)^k >\})$
            $C\{l\}\{index++\} \leftarrow ((X - \{< a \neq a_h, \widetilde{AT}(a) > \vee < a_h, \widetilde{A}(a_h)^k >\}^{\downarrow}, \widetilde{AT})$
          end
        end
      end
    end
  end
  $l++$
  until $|C\{l-1\}| = 0$;
end

method and program and then give the detail method of how to distinguish any two different objects (in different fuzzy concepts or incomplete fuzzy concepts) in the case study section.

5 Case study

It is worth noting that our multi-level cognitive concept learning method recognizes fuzzy concepts and incomplete fuzzy concepts on the basis of fuzzy membership values. Each concept contains two parts: objects set, attributes and the corresponding fuzzy membership values. The difference of fuzzy membership degrees directly determines the difference in the concepts that are recognized. The proposed method in Sect. 4 is mainly to achieve these three aims: to obtain all the fuzzy concepts or incomplete fuzzy concepts in data set; to recognize certain objects in each data set; and to distinguish between two objects (in different object sets of fuzzy concepts or incomplete fuzzy concepts). That is to say, the purpose of our method is to obtain all the useful information (about cognition) from a data set with fuzziness, instead of recognizing one certain concept.

In this section, we use an example to apply and understand our feature-oriented multi-level cognitive concept learning method in data sets with fuzziness. Also, the detailed method of how to distinguish between two different objects will be addressed. Moreover, a comparison with the existing *granular computing approach to two-way learning* (Xu and Li 2016) has been included to see the cognitive effect of these two methods.

Example 5.1 Figure 3 is a collection of seven common micro-expressions of a human being. Our goal is to recognize and distinguish between any two different micro-expressions by learning their concepts from these given seven micro-expressions in Fig. 3. And then we compare the fuzzy concepts recognized by using our multi-level method and two-way learning approach.

Figure 3 gives us an intuitive impression of micro-expressions, and then, we can get a collection of the facial features of these seven micro-expressions in Table 2. Table 2 gives us an intuitive facial features description of each of these micro-expressions.

In order to achieve the purpose of recognizing and distinguishing any two different micro-expressions, we need to translate the qualitative descriptions of these micro-expressions in Table 2 to quantitative descriptions. Without loss of generality, it is assumed that the baseline (normal value) is 0.5. For a certain micro-expression, if the corresponding eyebrows are higher than the normal value, then the membership degree of eyebrows is a fuzzy value in interval (0.5,1], otherwise it is in interval [0,0.5). Analogously, the baselines of the size of eyes, the size of mouth, the height of lips corner are 0.5, and all the membership degrees of these features are in interval [0,1]. According to this assumption, we get the quantitative description of the given seven micro-expressions in Table 3. In fact, Table 3 is a fuzzy information system formed by 7 kinds of micro-expressions

Fig. 3 A collection of seven human common micro-expressions

Table 2 Collection of facial features of micro-expressions

	Eyebrows	Eyes	Mouth	Lip corners
Happiness	Normal	Squinting	Slightly	Tipped up
Sadness	Tightened	Squinting	Close lightly	Pull down
Fear	Rise	Open	Slightly open	Normal
Anger	Drooping	Slightly open	Nervous	Normal
Disgust	Drooping	Slightly open	Tipped up	Rise
Surprise	Rise	Wide open	Open	Normal
Contempt	Slightly drooping	Slightly open	Slightly open	Rise

Table 3 Quantitative description of micro-expressions

	a_1	a_2	a_3	a_4
x_1	0.5	0.3	0.6	1
x_2	0.4	0.3	0.45	0.2
x_3	0.7	0.6	0.52	0.5
x_4	0.2	0.7	0.3	0.5
x_5	0.4	0.6	0.55	0.8
x_6	0.8	0.9	0.7	0.5
x_7	0.45	0.4	0.55	0.7

and 4 facial features where $U = \{x_1, x_2, \ldots, x_7\}$ is a set of common micro-expressions: "happiness," "sadness," "fear" "anger," "disgust," "surprise" and "contempt," respectively. The feature set is $AT = \{a_1, a_2, a_3, a_4\}$, which represents "eyebrows," "eyes," "mouth" and "lip corners," respectively.

In this section, by using our feature-oriented multi-level cognitive concept learning method in data sets with fuzziness which is investigated in the last section, we know all the objects in the fuzzy information system comprise U and AT. Then, we can compute all the fuzzy concepts and incomplete fuzzy concepts in each level. The corresponding cognitive concept maps based on objects and features can be produced directly.

By using our program *Algorithm* 1 of multi-level cognitive concept learning method in fuzzy data from the perspective of feature in Sect. 4, we can calculate all the fuzzy concepts and incomplete fuzzy concepts in this fuzzy information system.

For this fuzzy information system $\widetilde{S} = (U, AT, \widetilde{R})$, where $R : U \times AT \to [0, 1]$, all the fuzzy concepts and incomplete fuzzy concepts learned by feature-oriented multi-level cognitive concept learning method are listed as follows.

Level 1 $(U, \{<a_1, 0.2>, <a_2, 0.3>, <a_3, 0.3>, <a_4, 0.2>\})$ is a fuzzy concept, and it is obvious that $U^{\uparrow} = \widetilde{AT_1} = \{<a_1, 0.2>, <a_2, 0.3>, <a_3, 0.3>, <a_4, 0.2>\}$ is the fuzzy focal feature set.

Level 2 $(\{x_1, x_3, x_4, x_5, x_6, x_7\}, \{<a_1, 0.2>, <a_2, 0.3>, <a_3, 0.3>, <a_4, 0.5>\})$ is a fuzzy concept, and $(\{x_1, x_2, x_3, x_5, x_6, x_7\}, \{<a_1, 0.2>, <a_2, 0.3>, <a_3, 0.45>, <a_4, 0.2>\})$ is an incomplete fuzzy concept in this level.

Level 3 $(\{x_1, x_2, x_3, x_5, x_6, x_7\}, \{<a_1, 0.4>, <a_2, 0.3>, <a_3, 0.45>, <a_4, 0.2>\})$ and $(\{x_3, x_4, x_5, x_6, x_7\}, \{<a_1, 0.2>, <a_2, 0.4>, <a_3, 0.3>, <a_4, 0.5>\})$ are two complete fuzzy concepts. $(\{x_1, x_5, x_7\}, \{<a_1, 0.2>, <a_2, 0.3>, <a_3, 0.3>, <a_4, 0.7>\})$ is an incomplete fuzzy concept in this level.

Level 4 $(\{x_3, x_4, x_5, x_6\}, \{<a_1, 0.2>, <a_2, 0.6>, <a_3, 0.3>, <a_4, 0.5>\})$ and $(\{x_1, x_5, x_7\}, \{<a_1, 0.4>, <a_2, 0.3>, <a_3, 0.55>, <a_4, 0.7>\})$ are two complete fuzzy concepts. $(\{x_1, x_3, x_6, x_7\}, \{<a_1, 0.45>, <a_2, 0.3>, <a_3, 0.45>, <a_4, 0.2>\})$, $(\{x_2, x_5\}, \{<a_1, 0.4>, <a_2, 0.3>, <a_3, 0.45>, <a_4, 0.2>\})$ and $(\{x_5, x_7\}, \{<a_1, 0.2>, <a_2, 0.4>, <a_3, 0.3>, <a_4, 0.7>\})$ are all the three incomplete fuzzy concepts in this level.

Level 5 $(\{x_1, x_3, x_6, x_7\}, \{<a_1, 0.45>, <a_2, 0.3>, <a_3, 0.52>, <a_4, 0.5>\})$, $(\{x_4, x_6\}, \{<a_1, 0.2>, <a_2, 0.7>, <a_3, 0.3>, <a_4, 0.5>\})$ and $(\{x_5, x_7\}, \{<a_1, 0.4>, <a_2, 0.4>, <a_3, 0.55>, <a_4, 0.7>\})$ are all the three complete fuzzy concepts. $(\{x_5\}, \{<a_1, 0.4>, <a_2, 0.3>, <a_3, 0.45>, <a_4, 0.5>\})$, $(\{x_2\}, \{<a_1, 0.4>, <a_2, 0.3>, <a_3, 0.45>, <a_4, 0.2>\})$, $(\{x_3, x_5, x_6\}, \{<a_1, 0.2>, <a_2, 0.6>, <a_3, 0.45>, <a_4, 0.5>\})$ and $(\{x_1\}, \{<a_1, 0.4>, <a_2, 0.3>, <

Table 4 Number of concepts and incomplete concepts

	Level 1	Level 2	Level 3	Level 4	Level 5	Level 6	Level 7	Level 8	Level 9
NFC	1	1	2	2	3	5	2	1	0
NIFC	0	1	1	3	4	3	1	0	1
Total-F	1	2	3	5	7	8	3	1	1

$a_3, 0.6>, <a_4, 0.7>\})$ are four incomplete fuzzy concepts in this level.

Level 6 $(\{x_3, x_6, x_7\}, \{<a_1, 0.45>, <a_2, 0.4>, <a_3, 0.52>, <a_4, 0.5>\})$, $(\{x_5\}, \{<a_1, 0.4>, <a_2, 0.6>, <a_3, 0.55>, <a_4, 0.8>\})$, $(\{x_3, x_5, x_6\}, \{<a_1, 0.4>, <a_2, 0.6>, <a_3, 0.52>, <a_4, 0.5>\})$, $(\{x_7\}, \{<a_1, 0.45>, <a_2, 0.4>, <a_3, 0.55>, <a_4, 0.7>\})$ and $(\{x_1\}, \{<a_1, 0.5>, <a_2, 0.3>, <a_3, 0.6>, <a_4, 1>\})$ are all the five complete fuzzy concepts. $(\{x_1, x_7\}, \{<a_1, 0.45>, <a_2, 0.3>, <a_3, 0.52>, <a_4, 0.7>\})$, $(\{x_6\}, \{<a_1, 0.4>, <a_2, 0.7>, <a_3, 0.3>, <a_4, 0.5>\})$ and $(\{x_4\}, \{<a_1, 0.2>, <a_2, 0.7>, <a_3, 0.3>, <a_4, 0.5>\})$ are all the three incomplete fuzzy concepts in this level.

Level 7 $(\{x_1, x_7\}, \{<a_1, 0.45>, <a_2, 0.3>, <a_3, 0.55>, <a_4, 0.7>\})$ and $(\{x_6\}, \{<a_1, 0.8>, <a_2, 0.9>, <a_3, 0.7>, <a_4, 0.5>\})$ are all the two complete fuzzy concepts. $(\{x_3, x_6\}, \{<a_1, 0.5>, <a_2, 0.4>, <a_3, 0.52>, <a_4, 0.5>\})$ is the only incomplete fuzzy concept in this level.

Level 8 $(\{x_3, x_6\}, \{<a_1, 0.7>, <a_2, 0.6>, <a_3, 0.52>, <a_4, 0.5>\})$ is the only complete fuzzy concept in this level.

Level 9 $(\{x_3\}, \{<a_1, 0.7>, <a_2, 0.6>, <a_3, 0.52>, <a_4, 0.5>\})$ is the only incomplete fuzzy concept in this level.

Table 4 lists the number of all the fuzzy concepts and incomplete fuzzy concepts of each level in this fuzzy information system where *NFC*, *NIFC*, *total-F* mean the number of fuzzy concepts, the number of incomplete fuzzy concepts, the total number of fuzzy concepts and incomplete fuzzy concepts, respectively.

From Table 4, it is obvious that the trend of the change of the number of total (the number of fuzzy concepts and incomplete fuzzy concepts) in each level is increasing first and then decreasing. This is mainly caused by the main idea of our multi-level cognitive method.

For this fuzzy information system $\widetilde{S} = (U, AT, \widetilde{R})$, where $R : U \times AT \rightarrow [0, 1]$, we compute the necessary and sufficient fuzzy information granules (fuzzy concepts in our paper) by using the existing two-way learning approach (Xu and Li 2016).

Given an arbitrary fuzzy information granule $(\{x_1\}, \{<a_1, 0.1>, <a_2, 0.2>, <a_3, 0.3>, <a_4, 0.4>\})$, we get two necessary and sufficient fuzzy information granules (two fuzzy concepts) $(\{x_1\}, \{<a_1, 0.5>, <a_2, 0.3>, <a_3, 0.6>, <a_4, 1>\})$ and $(\{x_1, x_3, x_4, x_5, x_6, x_7\}, \{<a_1, 0.2>, <a_2, 0.3>, <a_3, 0.3>, <a_4, 0.5>\})$. We only recognize one object micro-expression *Happiness* (x_1).

Given an arbitrary fuzzy information granule $(\{x_2, x_4, x_6\}, \{<a_1, 0.3>, <a_2, 0.2>, <a_3, 0.4>, <a_4, 0.3>\})$, we get three necessary and sufficient fuzzy information granules (three fuzzy concepts) $(\{x_1, x_3, x_5, x_6, x_7\}, \{<a_1, 0.4>, <a_2, 0.3>, <a_3, 0.52>, <a_4, 0.5>\})$, $(\{x_6\}, \{<a_1, 0.8>, <a_2, 0.9>, <a_3, 0.7>, <a_4, 0.5>\})$ and $(U, \{<a_1, 0.2>, <a_2, 0.3>, <a_3, 0.3>, <a_4, 0.2>\})$. We only recognize one object micro-expression *Surprise* (x_6).

Since necessary and sufficient fuzzy information granule is obtained by training an arbitrary fuzzy information granule in Xu and Li (2016), different arbitrary fuzzy information granules will obtain different necessary and sufficient fuzzy information granules. While there are too many arbitrary fuzzy information granules for us to choose even in Table 3, we cannot list all possible results. So we only compare the results of our method and the above results based on two given arbitrary fuzzy information granules.

It is straightforward that we got 17 fuzzy concepts and recognized 7 micro-expressions by using our multi-level cognitive concept learning method, while we got up to 3 necessary and sufficient fuzzy information granules (fuzzy concepts in our paper) and recognized 1 micro-expression by using existing two-way learning approach. This clearly shows the advantages of our method of recognizing fuzzy concepts and recognizing objects. Although we may recognize all these 7 micro-expressions by setting different arbitrary fuzzy information granules, the difficulties are that there are too many choices to choose from and how to choose the right arbitrary information granule.

We have calculated all the fuzzy concepts and incomplete fuzzy concepts in each level. Then, we can map the corresponding figure based on the results mentioned above, which is shown in Fig. 4.

It is easy to see there is a different representation form of fuzzy set in Fig. 4. In other words, we use one representation of fuzzy set (e.g., $\{<a_1, 0.1>, <a_2, 0.2>, <a_3, 0.1>, <a_4, 0.2>, <a_5, 0.2>\}$) in our paper and use the other representation form of fuzzy set (e.g., $\{\frac{0.1}{a_1}, \frac{0.2}{a_2}, \frac{0.1}{a_3}, \frac{0.2}{a_4}, \frac{0.2}{a_5}\}$) in this figure to better demonstrate the cognitive results. Here,

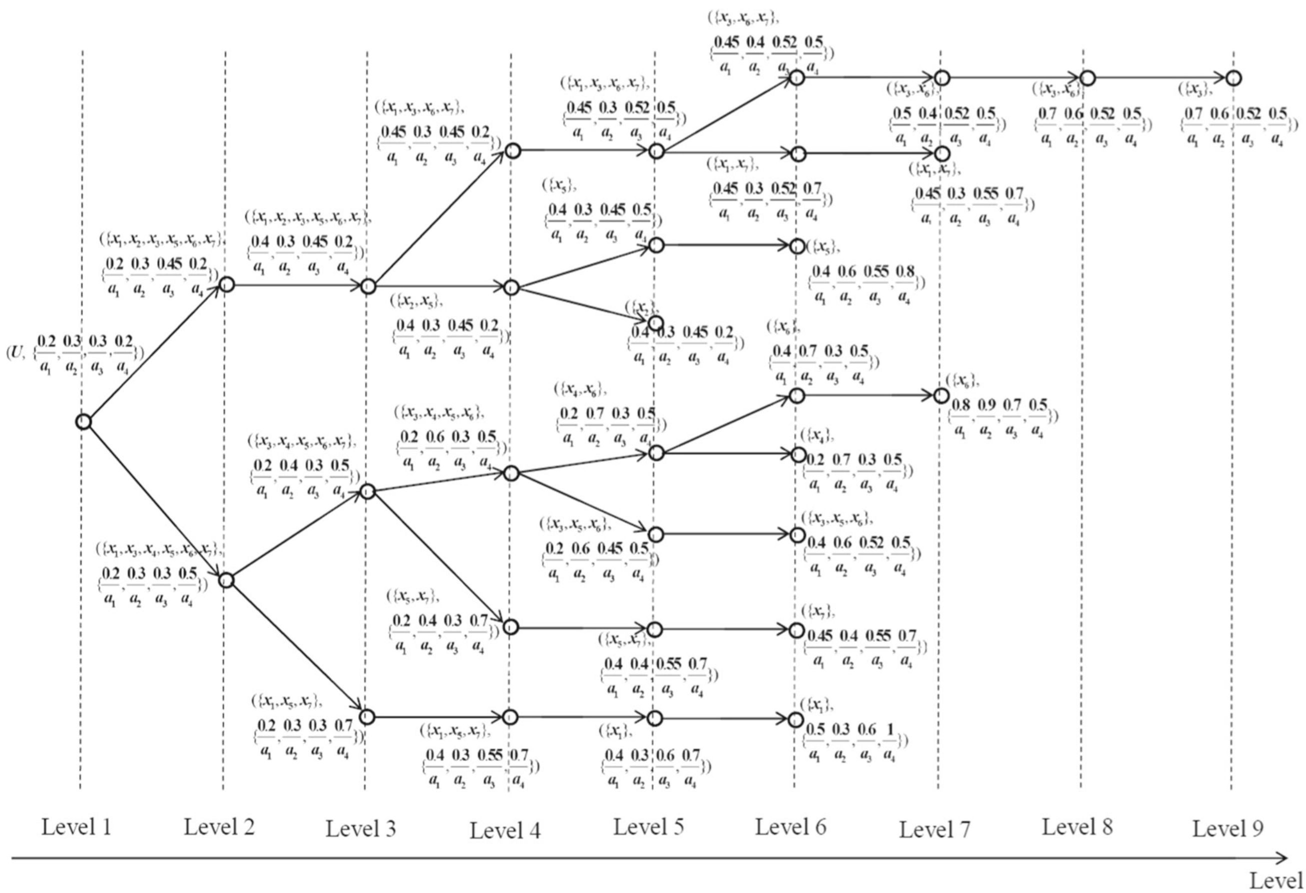

Fig. 4 Feature-oriented multi-level cognitive animals learning structure chart in fuzzy information system $\widetilde{S} = (U, AT, \widetilde{R})$

we take the fuzzy focal feature set as an example: $\widetilde{AT_1} = \{< a_1, 0.1 >, < a_2, 0.2 >, < a_3, 0.1 >, < a_4, 0.2 >, < a_5, 0.2 >\} = \{\frac{0.1}{a_1}, \frac{0.2}{a_2}, \frac{0.1}{a_3}, \frac{0.2}{a_4}, \frac{0.2}{a_5}\}$, where $\{< a_1, 0.1 >, < a_2, 0.2 >, < a_3, 0.1 >, < a_4, 0.2 >, < a_5, 0.2 >\}$ and $\{\frac{0.1}{a_1}, \frac{0.2}{a_2}, \frac{0.1}{a_3}, \frac{0.2}{a_4}, \frac{0.2}{a_5}\}$ are two different representations of fuzzy focal feature set $\widetilde{AT_1}$.

From Fig. 4, it is straightforward to see that the shallower the level, the larger the object set, the smaller one of the thresholds of all the features. The deeper the level, the smaller the object set, the larger one of the thresholds of all the features. That is to say, from the shallower level to the deeper level, there is an increase in one certain feature's threshold, while there is a decrease in objects which own more properties.

After cognitive all these seven micro-expressions, one may want to know how to distinguish any two different micro-expressions. So it is worth knowing how to distinguish any two objects (in different object sets of fuzzy concepts or incomplete fuzzy concepts) on the basis of the cognitive process we proposed. In order to distinguish any two different objects in a certain fuzzy information system, what one needs to do first is to outcrop these two objects, and then one can find a recent (incomplete) fuzzy concept straightforwardly through the cognitive path of these two objects, where these two objects are included in the recent (incomplete) fuzzy concept. At last, the classification criteria of the recent (incomplete) fuzzy concept can distinguish these two objects immediately. For example, if one wants to distinguish "sadness" (x_2) from "disgust" (x_5), first one can find these two micro-expressions in Fig. 4, and then a fuzzy concept $(\{x_2, x_5\}, \{< a_1, 0.4 >, < a_2, 0.3 >, < a_3, 0.45 >, < a_4, 0.2 >\})$ in Level 4 can be found. Then one can distinguish them based on the height of lips corner (a_1) according to the classification criteria of the fuzzy concept $(\{x_2, x_5\}, \{< a_1, 0.4 >, < a_2, 0.3 >, < a_3, 0.45 >, < a_4, 0.2 >\})$ in Level 4. If one pull down his or her lips corner (the height of lips corner is lower than the baseline (normal value) 0.5) in the micro-expression, then this micro-expression must be "sadness" (x_2), and the other micro-expression (lips corner raised up) is "disgust" (x_5); this is based on the incomplete fuzzy concepts $(\{x_5\}, \{< a_1, 0.4 >, < a_2, 0.3 >, < a_3, 0.45 >, < a_4, 0.5 >\})$ and $(\{x_2\}, \{< a_1, 0.4 >, < a_2, 0.3 >, < a_3, 0.45 >, < a_4, 0.2 >\})$ in Level 5. Then, we can distinguish two arbitrary objects (in different object sets of fuzzy concepts or incomplete fuzzy concepts) based on our feature-oriented multi-level cognitive concept learning method in fuzzy data.

Owing to the inadequate features in the fuzzy information system in Example 5.1, the corresponding fuzzy concepts and incomplete fuzzy concepts in the fuzzy information system $\widetilde{S} = (U, AT, \widetilde{R})$ are obtained. In order to make all the incomplete fuzzy concepts to be complete fuzzy concepts, one can achieve the fuzzy concepts by using the classical method (concept lattice and so on).

6 Experimental evaluation

For cognitive psychology, theory is the soul, model is the skeleton, and the experiments on the theory and model are flesh and blood. Therefore, the three parts are indispensable in the process of conceptual cognition. In this section, we will test our feature-oriented multi-level cognitive concept learning method in fuzzy data on some real-life data sets.

It is not difficult to find that the original data sets may not be a fuzzy information system to achieve the purpose of recognizing concepts. All kinds of information systems we may face could mainly be classical information system, fuzzy information system, interval value information system, fuzzy number information system, set value information system and so on.

For classical information system and fuzzy information system, we can directly use our feature-oriented multi-level cognitive concept learning method to achieve the goal of recognizing concepts. For interval value information system, to use our feature-oriented multi-level method in recognizing concepts, one of the methods to fuzzify the interval value information system is to translate all the feature values (which are interval values) to the median of interval values, and then we get a fuzzy information system. For fuzzy number information system, especially for the information system with qualitative descriptions, to use our method to recognize concepts, one must translate all the qualitative descriptions to quantitative descriptions. It is noteworthy that it will cause information loss problem, which has a significant negative impact on the results of the research if one translate this type of information systems to classical information systems. For set value information system, to preprocess this type of information systems, one can translate all the feature values (set values) to a number of intervals [0,1] by using the ratio of the cardinality of each set value to the cardinality of the union of all the set values of certain column. No matter which information system we use, it is difficult to recognize concepts directly from the information system, while our feature-oriented multi-level cognitive concept learning method can deal with pieces of information well by using some preprocess approaches. Our approaches reduce the influence of the information loss to a certain degree and cause the qualitative question to the appraisal quantitative one. So, our method can be used to solve the cognitive concept learning problems successfully for multiple information systems.

Table 5 Descriptions of testing date sets

Data set	Objects	Features
Letter recognition	8084	16
Vehicle	846	18
Wholesale customers	440	6
Wine quality-Red	1599	11
Wine quality-White	4898	11

To use our multi-level cognitive concept learning method in some real-life data sets, what we need to do first is the data preprocessing. We know that the relation matrix is a fuzzy matrix if the feature-oriented multi-level learning method can be applied to cognitive concepts. Therefore, the preprocessing of data sets is mainly to transform a certain relation matrix into a fuzzy matrix. And then one can use our multi-level method to recognize fuzzy concepts. In this paper, to ensure the relation matrix is a fuzzy matrix before applying our multi-level method, for any feature value, we convert the feature value to the ratio of this feature value to the maximum feature value of this feature(the maximum value in this column). Then, all the feature values can be changed into fuzzy values in interval [0,1], and the corresponding relation matrix is a fuzzy matrix. In this section, we apply our method to solve concept cognitive problem on five real-life data sets available from the UCI databases. The characteristics of these data sets are summarized in Table 5.

In order to test the program of Algorithm 1 proposed in Sect. 4, we will use these five data sets described in Table 5 to verify the validity of this program. By using the program of our feature-oriented multi-level cognitive concept learning method in fuzzy data, we can get all the fuzzy concepts and incomplete fuzzy concepts of these tested five data sets. The number of fuzzy concepts and incomplete fuzzy concepts in each level of these tested five data sets is listed in Tables 6, 7, 8, 9 and 10, where *NFC*, *NIFC*, *total-F* represent the number of fuzzy concepts, the number of incomplete fuzzy concepts, the total number of fuzzy concepts and incomplete fuzzy concepts, respectively. The numbers in heading represent the level numbers when one recognizes fuzzy concepts and incomplete fuzzy concepts in these five data sets.

From Tables 6, 7, 8, 9 and 10, we have got the following conclusions.

- At the beginning, the total number of recognized fuzzy concepts and incomplete fuzzy concepts increases by layers, and the total number of recognized fuzzy concepts and incomplete fuzzy concepts decreases by layers at a

Table 6 Letter recognition: number of concepts and incomplete concepts

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
NFC	0	0	1	2	2	2	2	4	3	9	12	13	17	27	31	43
NIFC	1	2	0	0	1	2	3	2	6	3	4	10	16	11	20	28
Total-F	1	2	1	2	3	4	5	6	9	12	16	23	33	38	51	71

	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
NFC	64	79	105	128	165	175	224	260	282	321	354	354	368	378	337	327
NIFC	38	57	69	94	94	131	139	165	176	196	191	221	224	190	190	153
Total-F	102	136	174	222	259	306	363	425	458	517	545	575	592	568	527	480

	33	34	35	36	37	38	39	40	41	42	43	44	45	46
NFC	300	257	223	172	135	113	80	46	32	19	12	6	3	2
NIFC	139	136	111	89	72	59	37	19	15	8	5	2	2	0
Total-F	439	393	334	261	207	172	117	65	47	27	17	8	5	2

Table 7 Vehicle: number of concepts and incomplete concepts

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
NFC	0	0	1	1	2	4	5	10	14	21	25	40	56	79
NIFC	1	2	0	1	1	1	4	3	7	9	18	16	29	39
Total-F	1	2	1	2	3	5	9	13	21	30	43	56	85	118

	15	16	17	18	19	20	21	22	23	24	25	26	27	28
NFC	110	139	167	196	210	222	245	239	241	230	214	214	219	217
NIFC	59	71	91	106	112	120	106	119	113	100	97	99	99	95
Total-F	169	210	258	302	322	342	351	358	354	330	311	313	318	312

	29	30	31	32	33	34	35	36	37	38	39
NFC	225	209	213	170	126	82	41	26	9	2	2
NIFC	89	111	90	76	47	27	11	2	1	1	0
Total-F	314	320	303	246	173	109	52	28	10	3	2

Table 8 Wholesale customers: number of fuzzy concepts and incomplete fuzzy concepts

	1	2	3	4	5	6	7	8	9
NFC	0	1	1	1	1	0	3	2	2
NIFC	1	1	1	1	1	4	0	4	0
Total-F	1	2	2	2	2	4	3	6	2

certain level. This is because some of cognitive paths have ended at some certain levels.

- After comparing Table 6 with Table 7, we find the number of levels in Table 6 is more than that in Table 7. This is mainly caused by the difference of these two data sets. The number of features of these two data sets is almost equal in *Letter recognition* and *Vehicle*, while *Letter recognition* data set has 8084 objects, and *Vehicle* has only 846 objects. In fact, Tables 9 and 10 have the similar results, but the difference is not too great. It is because the number of object set in Table 6 is 10 times of that in Table 7, while the number of object set in Table 10 is 3 times of that in Table 9. So we can draw a conclusion: The more the number of objects, the more the number of concept levels we obtained.

Next, we will make a comparative study in the number of fuzzy concepts which can be recognized by using our multi-level cognitive method with the existing granular computing approach to two-way learning (Xu and Li 2016). Since the authors only listed all the necessary and sufficient fuzzy information granules (i.e., the fuzzy concepts in our paper) in their experiment section, we then compare the number of fuzzy concepts using these two methods. Here, we only consider the maximum number of fuzzy concepts recognized in Xu and Li (2016). It means we use the maximum number comparing with us. Table 11 shows the comparison results of the number of fuzzy concepts based on feature-oriented

Table 9 Wine quality-Red: number of fuzzy concepts and incomplete fuzzy concepts

	1	2	3	4	5	6	7	8	9	10	11
NFC	0	0	1	2	4	7	13	16	19	27	33
NIFC	1	2	0	0	0	0	0	4	9	11	11
Total-F	1	2	1	2	4	7	13	20	28	38	44

	12	13	14	15	16	17	18	19	20	21
NFC	46	37	45	36	26	20	17	11	7	2
NIFC	16	38	31	34	25	22	23	8	5	0
Total-F	62	75	76	70	51	42	40	19	12	2

Table 10 Wine quality-White: number of fuzzy concepts and incomplete fuzzy concepts

	1	2	3	4	5	6	7	8	9	10	11	12
NFC	0	0	1	2	3	6	9	16	26	34	36	42
NIFC	1	2	0	0	0	0	0	0	2	10	19	18
Total-F	1	2	1	2	3	6	9	16	28	44	55	60

	13	14	15	16	17	18	19	20	21	22	23
NFC	40	48	41	52	56	52	35	26	7	6	2
NIFC	37	35	46	52	46	41	29	8	7	5	0
Total-F	77	83	87	104	102	93	64	34	14	11	2

multi-level cognitive concept learning method and two-way learning approach (Xu and Li 2016).

Based on the comparison results of multi-level method and two-way approach in Table 11, it is obvious that the number of fuzzy concepts recognized by using our method is larger than the two-way approach (Xu and Li 2016), especially in the data sets *Letter recognition* and *Vehicle*. This is mainly due to these two data sets having more objects and features than the other three data sets. This proves the validity of our multi-level method in conceptual cognition, especially the processing of the data set with a larger sets of objects and features. In addition, we can fully guide our further cognitive concept learning by using our feature-oriented multi-level cognitive concept learning method in fuzzy data. Another advantage of our multi-level approach is that by using the path of concept learning, we are able to have a better and long lasting memory of what we learned.

It is obvious that the more the number of fuzzy concepts we get, the more we know about the data set, the more effective the method used in cognitive concepts. Based on the results in Table 11, we find the effectiveness of our feature-oriented multi-level cognitive concept learning method with fuzzy data in cognitive concepts. Table 11 only lists the comparison of the number of fuzzy concepts obtained by those two different approaches. It is also worth noting that we got a large number of incomplete fuzzy concepts at the same time in the results of the tested five data sets which are listed in Tables 6, 7, 8, 9 and 10. The significance of the number of incomplete fuzzy concepts and the total number of fuzzy concepts and incomplete fuzzy concepts is what we should consider. Actually, we recognized some objects by getting some fuzzy concepts or incomplete fuzzy concepts in distinguishing two certain objects. Next, we will discuss the objects recognized by using our multi-level method. Xu and Li (2016) have not analyzed the recognized objects in their granular computing approach to two-way learning. So we only study the recognized objects by using our multi-level method. Table 12 lists the number of objects recognized by our feature-oriented multi-level cognitive concept learning method in the five tested data sets.

From Table 12, we can obtain the following conclusions.

- We can recognize parts of the objects in each data set, especially in the data set *Vehicle*, and we recognized 169 objects. This is mainly because that this data set has less objects and more features. So the less number of objects

Table 11 Comparison result of multi-level method and two-way approach (Xu and Li 2016)

	Data sets				
	Letter recognition	Vehicle	Wholesale customers	Wine quality-Red	Wine quality-White
Multi-level	5489	4226	11	369	540
Two-way	3	4	3	6	7

Table 12 The number of objects recognized by our multi-level method

	Data sets				
	Letter recognition	Vehicle	Wholesale customers	Wine quality-Red	Wine quality-White
Total objects	8084	846	440	1599	4898
Recognized objects	202	169	5	46	51

and the more number of features, the more objects we can recognize.

- This explains the validity of our method in recognizing the nature of objects, while in Xu and Li (2016), the authors cannot recognize the objects in the data set. This further illustrates that our method is more useful in practice.

In fact, the aims of our feature-oriented multi-level cognitive concept learning method in fuzzy data are: (1) to obtain all the fuzzy concepts and incomplete fuzzy concepts in data set; (2) to recognize certain objects in each data set; (3) to distinguish between two objects (in different object sets of fuzzy concepts or incomplete fuzzy concepts). Table 11 shows the advantage of our method in achieving aim (1). Table 12 shows the superiority of our method in achieving aim (2). As for the aim (3), it is difficult to directly explain how to distinguish between two different objects, because the objects are too many for us to distinguish. The specific approach of distinguishing two certain objects (in different object sets of fuzzy concepts or incomplete fuzzy concepts) can be found in the case study section.

7 Conclusions

In this paper, by defining a special fuzzy feature set: fuzzy focal feature set, we first simulated human cognitive process and mechanism, and then, we established a feature-oriented multi-level cognitive concept learning structure based on the human cognitive psychology and philosophy. Furthermore, an algorithm of cognitive concepts learning is studied and established in a fuzzy information system. Finally, to interpret and help understand our method of conceptual cognition, we conducted an experiment about micro-expressions cognition, and then, we map the feature-oriented multi-level cognitive concept learning figure. Moreover, the skill of how to distinguish any two objects (in different object set of fuzzy concepts or incomplete fuzzy concepts) based on our multi-level method is introduced. In addition, five data sets from UCI databases are tested to verify the effectiveness of our feature-oriented multi-level cognitive concept learning method in data sets with fuzziness and the corresponding program. Compared with the existing granular computing approach to two-way learning, results have shown the feasibility and superiority of our proposed method. In the future, we will consider the structure of object-oriented multi-level cognitive concept learning method in fuzzy data from the perspective of objects, and then a feature-object-oriented multi-level cognitive concept learning pattern will be taken into account to solve the problem of cognitive concept learning from both features and objects points of views (i.e., the whole fuzzy information system).

Acknowledgements This work is supported by the Macau Science and Technology Development Fund (No. 081/2015/A3), the National Natural Science Foundation of China (Nos. 71471060, 61472463, 61402064, and 61772002), the Science and Technology Research Program of Chongqing Municipal Education Commission (Grant No. KJ1709221).

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

Informed consent Informed consent was obtained from all individual participants included in the study.

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